Binomial Theorem

→ $(x+y)^2 = x^2 + 2xy + y^2$

$(x+y)(x+y) = {}^2C_2 x^2 y^0 + {}^2C_1 x y^1 + {}^2C_0 x^0 y^2$

$(x+y)^2 = {}^2C_0 y^2 x^0 + {}^2C_1 x^1 y^1 + {}^2C_2 x^2 y^0$

→ $(x+y)^3 = (x+y)(x+y)(x+y) = {}^3C_3 x^3 y^0 + {}^3C_2 x^2 y^1 + {}^3C_1 x y^2 + {}^3C_0 x^0 y^3$

$= {}^3C_0 x^3 y^0 + {}^3C_1 x^2 y^1 + {}^3C_2 x^1 y^2 + {}^3C_3 x^0 y^3$

→ $(x+y)^4 = {}^4C_0 x^4 y^0 + {}^4C_1 x^3 y^1 + {}^4C_2 x^2 y^2 + {}^4C_3 x^1 y^3 + {}^4C_4 x^0 y^4$

Standard Binomial Expansion ($n \in W$)

$$(ax+by)^n = {}^nC_0 a^n x^n b^0 y^0 + {}^nC_1 a^{n-1} b^1 x^{n-1} y^1 + {}^nC_2 a^{n-2} b^2 x^{n-2} y^2 + \cdots + {}^nC_r a^{n-r} b^r x^{n-r} y^r + \cdots + {}^nC_n a^0 b^n x^0 y^n .$$

$$(ax+by)^n = \underbrace{\sum_{r=0}^{n} {}^nC_r a^{n-r} b^r x^{n-r} y^r}_{(r+1)^{th}\ \text{term}} \quad *$$

Total No. of terms $= n+1$

$$\boxed{\text{General term} = T_{r+1} = {}^nC_r a^{n-r} b^r x^{n-r} y^r}$$

${}^nC_r \longrightarrow$ Binomial Coefficient

↓

for Properties refer to P & C.

$$\boxed{{}^nC_r a^{n-r} b^r \longrightarrow \text{Term Coefficient}}$$

$$(2x+3y)^5 = {}^5C_0\, 2^5 3^0 x^5 y^0 + {}^5C_1\, 2^4 3^1 x^4 y^1 + {}^5C_2\, 2^3 3^2 x^3 y^2 + {}^5C_3\, 2^2 3^3 x^2 y^3 + {}^5C_4\, 2^1 3^4 x^1 y^4 + {}^5C_5\, 2^0 3^5 x^0 y^5.$$

$$\left(\tfrac{1}{3}x - \tfrac{1}{2}y\right)^{10} = {}^{10}C_0 \left(\tfrac{1}{3}\right)^{10}\left(-\tfrac{1}{2}\right)^0 x^{10} y^0 + {}^{10}C_1 \left(\tfrac{1}{3}\right)^9 \left(-\tfrac{1}{2}\right)^1 x^9 y^1 + {}^{10}C_2 \left(\tfrac{1}{3}\right)^8 \left(-\tfrac{1}{2}\right)^2 x^8 y^2 + \dots \dots + {}^{10}C_{10} \left(\tfrac{1}{3}\right)^0 \left(-\tfrac{1}{2}\right)^{10} x^0 y^{10}$$

Some Standard Expansion:

1. $(x+y)^n = {}^nC_0 x^n y^0 + {}^nC_1 x^{n-1} y^1 + {}^nC_2 x^{n-2} y^2 + \dots\dots + {}^nC_n x^0 y^n$
   $= \sum_{r=0}^{n} {}^nC_r x^{n-r} y^r.$

2. $(x-y)^n = {}^nC_0 x^n y^0 - {}^nC_1 x^{n-1} y^1 + {}^nC_2 x^{n-2} y^2 + \dots\dots + {}^nC_n x^0 (-1)^n y^n$
   $= \sum_{r=0}^{n} (-1)^r\, {}^nC_r x^{n-r} y^r$

3. $(x+y)^n + (x-y)^n = 2\left[{}^nC_0 x^n y^0 + {}^nC_2 x^{n-2} y^2 + {}^nC_4 x^{n-4} y^4 + \dots\dots\right]$

4. $(x+y)^n - (x-y)^n = 2\left[{}^nC_1 x^{n-1} y^1 + {}^nC_3 x^{n-3} y^3 + {}^nC_5 x^{n-5} y^5 + \dots\dots\right]$

** 5. $f(x) = (1+x)^n = {}^nC_0 + {}^nC_1 x^1 + {}^nC_2 x^2 + {}^nC_3 x^3 + \dots\dots + {}^nC_n x^n = \sum_{r=0}^{n} {}^nC_r x^r$

** 6. $f(-x) = (1-x)^n = {}^nC_0 - {}^nC_1 x^1 + {}^nC_2 x^2 - {}^nC_3 x^3 + \dots\dots + (-1)^n\, {}^nC_n x^n = \sum_{r=0}^{n} (-1)^r\, {}^nC_r x^r$

Note:

1. $f(1) = {}^nC_0 + {}^nC_1 + {}^nC_2 + \dots\dots + {}^nC_n = 2^n$
   $f(1) = \sum_{r=0}^{n} {}^nC_r$

2. $f(-1) = 0 = {}^nC_0 - {}^nC_1 + {}^nC_2 - {}^nC_3 + \dots\dots$
   $= \sum_{r=0}^{n} (-1)^r\, {}^nC_r$

$f'(1) = n2^{n-1} = \sum_{r=0}^{n} r \cdot {}^nC_r$

$f'(-1) = 0 = \sum_{r=0}^{n} (-1)^r\, r\, {}^nC_r$

**.

3. $f(1)+f(-1) = 2^{n} = 2\left[{}^{n}C_{0}+{}^{n}C_{2}+{}^{n}C_{4}\ldots\ldots\right]$

$${}^{n}C_{0}+{}^{n}C_{2}+{}^{n}C_{4}+\ldots\ldots\ldots = 2^{n-1}$$

**.

4. $f(1)-f(-1) = 2^{n} = 2\left[{}^{n}C_{1}+{}^{n}C_{3}+{}^{n}C_{5}+\ldots\ldots\right]$

$${}^{n}C_{1}+{}^{n}C_{3}+{}^{n}C_{5}+\ldots\ldots = 2^{n-1}$$

5. $f(x) = (1+x)^{n} = {}^{n}C_{0}+{}^{n}C_{1}x^{1}+{}^{n}C_{2}x^{2}+\ldots\ldots\ldots+{}^{n}C_{n}x^{n}$

Diff. w.r.t x

$$f'(x) = n(1+x)^{n-1} = 0+1.{}^{n}C_{1}x^{0}+2.{}^{n}C_{2}x^{1}+\ldots\ldots+n.{}^{n}C_{n}x^{n-1}$$

$$\boxed{f'(1) = n.2^{n-1} = 1.{}^{n}C_{1}+2.{}^{n}C_{2}+\ldots\ldots\ldots+n.{}^{n}C_{n} = \sum_{r=0}^{n} r.{}^{n}C_{r}}$$ **.

$$xf'(x) = n\underbrace{(1+x)^{n-1}x}_{\text{diff. w.r.t } x} = 1.{}^{n}C_{1}x^{1}+2.{}^{n}C_{2}x^{2}+3.{}^{n}C_{3}x^{3}+\ldots\ldots+n.{}^{n}C_{n}x^{n}$$

$$n\left[1.(1+x)^{n-1}+x.(n-1)(1+x)^{n-2}\right] = 1^{2}.{}^{n}C_{1}x^{0}+2^{2}\,{}^{n}C_{2}x^{1}+3^{2}\,{}^{n}C_{3}x^{2}+\ldots\ldots+n^{2}\,{}^{n}C_{n}x^{n-1}$$

$x=1$

$$\boxed{n\left(2^{n-1}+(n-1)2^{(n-2)}\right) = 1^{2}.{}^{n}C_{1}+2^{2}\,{}^{n}C_{2}+3^{2}\,{}^{n}C_{3}+\ldots\ldots+n^{2}\,{}^{n}C_{n}}$$

$$n.2^{n-1}+n(n-1)2^{n-2} = \sum_{r=0}^{n} r^{2}\,{}^{n}C_{r}$$

6. $\frac{{}^{n}C_{0}}{1}+\frac{{}^{n}C_{1}}{2}+\frac{{}^{n}C_{2}}{3}+\ldots\ldots+\frac{{}^{n}C_{n}}{n+1} = \frac{C_{0}}{1}+\frac{C_{1}}{2}+\frac{C_{2}}{3}+\ldots+\frac{C_{n}}{n+1} = ?$

$$(1+x)^{n} = {}^{n}C_{0}+{}^{n}C_{1}x^{1}+{}^{n}C_{2}x^{2}+\ldots\ldots\ldots+{}^{n}C_{n}x^{n}$$

Integrate w.r.t x

$$\int_{0}^{x}(1+x)^{n}dx = \int_{0}^{x}{}^{n}C_{0}dx+\int_{0}^{x}{}^{n}C_{1}x^{1}dx+\int_{0}^{x}{}^{n}C_{2}x^{2}dx+\ldots\ldots+\int_{0}^{x}{}^{n}C_{n}x^{n}dx$$

$$\Rightarrow \left[\frac{(x+1)^{n+1}}{n+1}\right]_{0}^{x} = \left[\frac{{}^{n}C_{0}}{1}x^{1}+\frac{{}^{n}C_{1}}{2}x^{2}+\frac{{}^{n}C_{2}}{3}x^{3}+\ldots\ldots+\frac{{}^{n}C_{n}}{n+1}x^{n+1}\right]$$

$x=1$

$$\Rightarrow \frac{2^{n+1}}{n+1} - \frac{1}{n+1} = \frac{{}^{n}C_0}{1} + \frac{{}^{n}C_1}{2} + \frac{{}^{n}C_2}{3} + \cdots\cdots + \frac{{}^{n}C_n}{n+1}$$

Use of General term $(n \in I^+)$

$$(ax+by)^n = \sum_{r=0}^{n} \underbrace{{}^{n}C_r\, a^{n-r} b^r}_{\text{Term Coeff.}}\; x^{n-r} y^r$$

$$\underbrace{T_{r+1} = {}^{n}C_r\, a^{n-r}\, b^r\, x^{n-r}\, y^r}_{\text{General term}}$$

1. ↳ To find any term in the Expansion.

Ques. find 13th term in the Expansion of $\left(9x - \frac{1}{3\sqrt{x}}\right)^{18}$, $x \neq 0$

General term $T_{r+1} = {}^{18}C_r\, 9^{18-r} \left(-\frac{1}{3}\right)^r x^{18-r} \left(\frac{1}{\sqrt{x}}\right)^r$

$$T_{r+1} = {}^{18}C_r\; 3^{36-2r} (-1)^r\, 3^{-r}\, x^{18-r}\, x^{-r/2}$$

$$T_{r+1} = (-1)^r\; {}^{18}C_r\; 3^{(36-3r)}\, x^{\left(18-\frac{3r}{2}\right)}$$

$r=12$

$$T_{13} = (-1)^{12}\; {}^{18}C_{12}\; 3^{(36-3\times12)}\, x^{\left(18-\frac{3\times12}{2}\right)}$$

$$= {}^{8}C_{12}\, 3^0 x^0$$

$$= {}^{8}C_{12}$$

Ques. Find a, band n in the expanion of $(a+b)^n$ if the first three terms of the expansion are 729, 7290 and 30375, respectivly.

$$T_{r+1} = {}^nC_r\, a^{n-r} b^r$$

$r=0 \quad T_1 = {}^nC_0 a^n = 729 \Rightarrow a^n = 729$ — ①

$r=1 \quad T_2 = {}^nC_1 a^{n-1} b^1 = n\, a^n \frac{b}{a} = 7290$ — ⑪ $= n \times \cancel{729} \times \frac{b}{a} = \underset{10}{\cancel{7290}}$

$\frac{nb}{a} = 10$ — ⑪

$r=2 \quad T_3 = {}^nC_2 \cdot a^{n-2} b^2 = 30375$

$$\frac{n(n-1)}{2} a^n \frac{b^2}{a^2} = 30375$$

$$\left(\frac{nb}{a}\right) \cdot \left(\frac{n-1}{2n}\right) a^n \left(\frac{nb}{a}\right) = 30375$$

$$10 \times \frac{n-1}{2n} \times 729 \times 10 = 30375$$

$$\frac{n-1}{2n} \times \frac{30375}{729 \times 10 \times 10}$$

$$\frac{n-1}{n} = \frac{5}{6}$$

$$\frac{n-1}{n} = \frac{6-1}{6}$$

$n=6 \Rightarrow \frac{{}^3\cancel{6} \times b}{a} = \cancel{10}^5$

$\Rightarrow 3b/a = 5 \quad$ also $a^6 = 729$

$3b = 5a \qquad a^6 = (3)^6$

$a = 3$

$b = 5$

$$\therefore \left.\begin{matrix} a=3 \\ b=5 \\ n=6 \end{matrix}\right\}$$

2. To find term Coefficient in any Expansion

Find the term Coefficient of 14th term in the expansion of $(3x-4y)^{20}$

$= T_{r+1} = {}^{20}C_r\, 3^{20-r} (-4)^r x^{(20-r)} y^r$

$$T_{r+1} = \underbrace{{}^{20}C_r\, 3^{(20-r)} (-4)^r}_{\Downarrow} x^{(20-r)} y^r$$

Term Coefficient

$r = 13$

$T_{14} = {}^{20}C_{13}\, 3^7 (-4)^{13} x^7 y^{13}$

Coeff. of $T_{14} = -{}^{20}C_{13}\, 3^7\, 4^{13}$.

3. To find Coefficient of any Power of variable in the expansion

Ques: Find Coefficient x^{13} in the expansion of $\left(\frac{1}{2} - \frac{3}{4}x\right)^{20}$

General term

$\Rightarrow T_{r+1} = {}^{20}C_r \left(\frac{1}{2}\right)^{20-r} \left(-\frac{3}{4}x\right)^r$

$T_{r+1} = (-1)^r\, {}^{20}C_r \left(\frac{1}{2}\right)^{(20-r)} \left(\frac{3}{4}\right)^r x^r$

To get coefficient of x^{13}, $r = 13$

$$T_{14} = (-1)^{13}\, {}^{20}C_{13} \left(\frac{1}{2}\right)^7 \left(\frac{3}{4}\right)^{13} x^{13}$$

4. To find term independent of x (independent ~~of~~ variable)

Ques. find the term free from x in the Expansion of $\left(\frac{1}{2}x - \frac{1}{3x}\right)^{19}$

$$T_{r+1} = {}^{19}C_r \left(\frac{1}{2}\right)^{19-r}\left(\frac{-1}{3}\right)^r x^{19-r}\left(\frac{1}{x}\right)^r$$

$$= (-1)^r\, {}^{19}C_r \left(\frac{1}{2}\right)^{19-r}\left(\frac{1}{3}\right)^r x^{(19-2r)}$$

To get term independent of x

$19-2r=0$

$r = 19/2$ (Not possible)

$r \in W$

$\therefore$ No term independent of x.

Ques. Find the term free from x in the Expansion of $\left(\frac{9}{x} - \frac{x^2}{5}\right)^6$

$$T_{r+1} = {}^6C_r (9)^{6-r}\left(-\frac{1}{5}\right)^r\left(\frac{1}{x}\right)^{6-r}(x^2)^r$$

$$= (-1)^r\, {}^6C_r\, 9^{6-r} 5^{-r} x^{-6+r} x^{2r}$$

$$= (-1)^r\, {}^6C_r\, 9^{6-r} 5^{-r} x^{(3r-6)}$$

$3r-6=0$

$r=2$

$T_3 = (-1)^2\, {}^6C_2\, 9^4 5^{-2} x^0$

$T_3 = {}^6C_2 \frac{9^4}{5^2}$ (Term independent of x)

5. To find term free from radicals or No. of rational terms in the expansion.

fractional term Power Pr nhi hoga.

Que. find no. of Rational term in the expansion of $(\sqrt{3}+\sqrt{2})^{10}$

$$T_{r+1} = {}^{10}C_r \left(\sqrt{3}\right)^{(10-r)} \left(\sqrt{2}\right)^{r}$$

$$= {}^{10}C_3 \left(\frac{10-r}{2}\right) 2^{(r/2)}$$

r	$\frac{10-r}{2}$	$r/2 \Rightarrow$ Integer	
0	✓	✓	$\longrightarrow T_1$
2	✓	✓	$\longrightarrow T_3$
4	✓	✓	$\longrightarrow T_5$
6	✓	✓	$\longrightarrow T_7$
8	✓	✓	$\longrightarrow T_9$
10	✓	✓	$\longrightarrow T_{11}$

1st, 3rd 5th, 7th, 9th, & 11th terms are rational terms

No. of rational terms = 6

Que. How many terms are free from radical signs in the expansion of $\left(y^{1/5}+x^{1/10}\right)^{55}$.

$$= \left(y^{1/5}+x^{1/10}\right)^{55}$$

$$T_{r+1} = {}^{55}C_r \, y^{\left(\frac{55-r}{5}\right)} x^{\left(r/10\right)}$$

r	$\frac{55-r}{5}$	$r/10$	
$r=0$	✓	✓	T_1
$r=10$	✓	✓	T_{11}
$r=20$	✓	✓	T_{21}
$r=30$	✓	✓	T_{31}
$r=40$	✓	✓	T_{41}
$r=50$	✓	✓	T_{51}

Radical free terms.

6. To find middle term in any expansion.

$$(ax+by)^n = \sum_{r=0}^{n} {}^nC_r\, a^{n-r} b^r x^{n-r} y^r$$

if $n =$ even

No. of term $= n+1 \longrightarrow$ odd

There will one middle term

i.e $\left(\frac{n+1+1}{2}\right)^{th} = \left(\frac{n}{2}+1\right)^{th}$ term.

$$\text{Middle term} = T_{\left(\frac{n}{2}+1\right)} = \underbrace{{}^nC_{n/2}\, a^{n/2}\, b^{n/2}}_{}\, x^{n/2}\, y^{n/2}$$

$$\boxed{r = n/2}$$

if $n =$ odd

No. of term $= n+1$ —— even

There will be two middle term

i.e $\left(\frac{n+1}{2}\right)^{th}$ & $\left(\frac{n+3}{2}\right)^{th}$ terms.

↓

$r = \frac{n-1}{2}$ & $\frac{n+1}{2}$

$$T_{\left(\frac{n+1}{2}\right)} = T_{\left(\frac{n-1}{2}\right)+1} = {}^nC_{\frac{n-1}{2}}\, a^{\left(n-\frac{n-1}{2}\right)}\, b^{\frac{n-1}{2}}\, x^{\left(n-\frac{n-1}{2}\right)}\, y^{\left(\frac{n-1}{2}\right)}$$

$$T_{\left(\frac{n+3}{2}\right)} = T_{\left(\frac{n+1}{2}+1\right)} = {}^nC_{\frac{n+1}{2}}\, a^{\left(n-\frac{n+1}{2}\right)}\, b^{\left(\frac{n+1}{2}\right)}\, x^{\left(n-\frac{n+1}{2}\right)}\, y^{\left(\frac{n+1}{2}\right)}$$

Note:

$$(1+x)^n = {}^nC_0 + {}^nC_1 x + {}^nC_2 x^2 + {}^nC_3 x^3 + \dots\dots + {}^nC_n x^n = \sum_{r=0}^{n} {}^nC_r x^r$$

Middle term

→ n - even , n+1 = odd

$$\text{Middle terms} = T_{\left(\frac{n+1+1}{2}\right)} = T_{\left(\frac{n}{2}+1\right)} = \boxed{{}^nC_{n/2}}\, x^{n/2}$$

→ n-odd , n+1 = even

$$\text{Middle terms} = T_{\left(\frac{n+1}{2}\right)} = T_{\left(\frac{n-1}{2}\right)+1} = {}^nC_{\frac{n-1}{2}}\, x^{\frac{n-1}{2}}$$

or,

$$T_{\left(\frac{n+3}{2}\right)} = T_{\left(\frac{n+1}{2}\right)+1} = {}^nC_{\frac{n+1}{2}}\, x^{\frac{n+1}{2}}$$

Ques. find middle term in $(1+x)^{2n}$, $n \in \omega$.

$(1+x)^{2n}$

$2n$ — even , $2n+1$ — odd

$$\text{Middle term} = \left(T_{\frac{2n}{2}+1}\right) = T_{n+1} = {}^{2n}C_n\, x^n$$

$$= \frac{2n!}{n!\,n!}\, x^n$$

7. To find the term having Greatest Coefficient or Greatest term.

$n \in W$

$$(ax+by)^n = \sum_{r=0}^{n} \underbrace{{}^nC_r\, a^{n-r} b^r x^{n-r} y^r}_{\Downarrow \; T_{r+1}}$$

$$\begin{tikzcd}
 & & \text{Greatest binomial Coefficient} \arrow[dddr, dashed, no head] & \\
 & {}^nC_2 \arrow[ur] & & \\
 {}^nC_1 \arrow[ur] & & & \\
 {}^nC_0 \arrow[u] & & & {}^nC_{n-1} \\
 & & & {}^nC_n
\end{tikzcd}$$

$$= T_1 + T_2 + \dots\dots + T_{r-1} + T_r + T_{r+1} + T_{r+2} + \dots\dots\dots + T_n$$

$$T_r < T_{r+1} > T_{r+2}$$

Only for term having greatest Coefficient.

$$\left|\frac{n|b|-|a|}{|a|+|b|}\right| \le r \le \left|\frac{n|b|-|a|}{|a|+|b|}\right| + 1$$

$r \in W$

T_{r+1} will have greatest Coefficient.

for Greatest term

$$\left|\frac{n|b\,y|-|a\,x|}{|a\,x|+|b\,y|}\right| \le r \le \left|\frac{n|b\,y|-|a\,x|}{|a\,x|+|b\,y|}\right| + 1$$

Ques. Find the numerically greatest term in expansion $(3-5x)^{15}$ when $x = 1/5$.

$(3-5x)^{15}$ at $x = 1/5$

$$\frac{n|by|-|ax|}{|ax|+|by|} = \frac{15|(-5x)|-|3|}{|3|+|-5x|}$$

$$= \frac{15\times 1-3}{3+1}$$

$$= \frac{12}{4} = 3$$

$3 \le r \le 4$ ∴ T_4 & T_5 will have numerically greatest value.

$r = 3, 4$

Ques $\left(1+\frac{2}{3}x\right)^{15}$ find greatest Coefficient,

$$\frac{n|b|-|a|}{|a|+|b|} = \frac{\overset{5}{\cancel{15}} \times 2/\cancel{3} - 1}{2/3+1}$$

$$= 9/\tfrac{5}{3}$$

$$= \frac{27}{5} = 5.4$$

$5.4 \le r \le 5.4 + 1$

$5.4 \le r \le 6.4$

$\boxed{r = 6}$

∴ T_7 will have greatest Coefficient.

Q. Given, that 4th term in the expansion of $\left(2+\frac{3}{8}x\right)^{10}$ has the maximum numerical value. Find the range of values of x.

$$\left(2+\frac{3}{8}x\right)^{10}$$

$$T_{r+1} = {}^{10}C_r\, 2^{10-r} (3/8)^r x^r$$

$$T_3 < T_4 > T_5$$

$$= {}^{10}C_2\, 2^8 \left(\frac{3}{8}\right)^2 x^2 < {}^{10}C_3\, 2^7 (3/8)^3 x^3 > {}^{10}C_4\, 2^6 (3/8)^4 x^4$$

$$= 2^2 \times {}^{10}C_2 < {}^{10}C_3 \times 2 \times 3/8 \times x > {}^{10}C_4 (3/8)^2 x^2$$

$$\frac{\cancel{2^2} \times \frac{10!}{8!2!}}{\frac{10!}{3!7!} \times 2} \times \frac{8}{3} < x$$

$$2 \times \frac{3}{8} \times \frac{8}{3} < x$$

$$2 < x$$

$$\frac{\cancel{10!}}{3!7!} \times 2 \times \frac{3}{8} x > \frac{10!}{6!4!} (3/8)^2 x^2$$

$$\frac{1}{7} \times 2 \times x > \frac{1}{4} \times \frac{3}{8} x^2$$

$$0 > \frac{3}{32} x^2 - \frac{2}{7} x$$

$$0 > \frac{(21x-64)x}{32 \times 7}$$

$$x \in \left(0, \frac{64}{21}\right)$$

$$\cap$$

$$x \in \left(2, \frac{64}{21}\right)$$

method ②

$$\frac{nb-a}{a+b} \le 3 \le \frac{nb-a}{a+b} + 1$$

$$\frac{10 \times \frac{3}{8} x - 2}{2 + \frac{3}{8} x} \le 3 \le \frac{\frac{10\times3}{8} x - 2}{2 + \frac{3}{8} x} + 1$$

$$\frac{30x-16}{16+3x} \le 3 \quad \& \quad 2 \le \frac{30x-16}{16+3x}$$

$$\frac{30x-16-48-9x}{16+3x} \le 0$$

$$\& \quad \frac{30x-16-32-6x}{16+3x}$$

$$\frac{21x-64}{3x+16} \le 0 \quad \& \quad 0 \le \frac{24x-48}{3x+16}$$

$$\Downarrow$$

$$x \in \left[2, \frac{64}{21}\right]$$

if 4th term have maximum numerical value

then

$$x \in \left[\frac{-64}{21}, -2\right] \cup \left[2, \frac{64}{21}\right].$$

Ques. Find the algebraically second largest term in the expansion of $(3-2x)^{15}$ at $x = 4/3$.

$$(1+x)^n = {}^nC_0 + {}^nC_1 x^1 + {}^nC_2 x^2 + \dots + {}^nC_n x^n$$

$$T_{r+1} = {}^nC_r x^r$$

T_{r+1} = Greatest term

Booklet method.

$$T_r < T_{r+1} > T_{r+2}$$

$$T_r \le T_{r+1}$$

Guruji's

$$\frac{n|by| - |ax|}{|ax| + |by|}$$

$$= \frac{\overset{5}{\cancel{15}} \times 2 \times 4/\cancel{3} - 3}{3 + 2 \times 4/3}$$

$$= \frac{137..\times 3}{9+8}$$

$$= \frac{37 \times 3}{17}$$

$$= \frac{111}{17} = 6.5$$

$$6.5 \le r \le 7.5$$

$$r = 7$$

T_8 = greatest term

$$T_7 = {}^{15}C_6\, 3^{15-6}(-2)^6\left(\frac{4}{3}\right)^6$$

$$T_9 = {}^{15}C_8\, 3^{15-8}(-2)^8\left(\frac{4}{3}\right)^8$$

$$T_7 = \frac{15!}{6!\,9!}\, 3^9\, 2^6 \left(\frac{4}{3}\right)^6$$

$$= \frac{\frac{8!\,7!}{6!\,9!} \cdot \frac{3^2}{2^2}\; {}^{15}C_8\, 3^7\, 2^8 \left(\frac{4}{3}\right)^8}{\left(\frac{4}{3}\right)^2}$$

$$T_7 = \frac{7}{\cancel{9}} \times \left(\frac{3}{4}\right)^2 \times \frac{\cancel{3^2}}{2^2} T_9$$

$$T_7 = \frac{7 \times 9}{16 \times 4} T_9$$

$$T_7 = \frac{63}{64} T_9$$

$$\underset{\overset{\uparrow}{\text{Greatest algebraically}}}{T_9} > \underset{\hookrightarrow \text{Second largest.}}{T_7}$$

Ques. Find the greatest term in the expansion of $(1+4x)^8$ at $x = \frac{1}{3}$

$$\frac{n|b| - |a|}{|b| + |a|} = \frac{8 \times \frac{4}{3} - 1}{\frac{4}{3} + 1} = \frac{29}{7}$$

$$\frac{n|b| - |a|}{|b| + |a|} = \frac{29}{7} = 4.14$$

$$4.14 \leq r \leq 4.14 + 1$$

$$4.14 \leq r \leq 5.14 \quad r \in W$$

$$\therefore r = 5$$

$\therefore T_6$ will be greatest term.

$$T_6 = {}^8C_5 (4x)^5$$

$$= {}^8C_5\, 4^5 \left(\frac{1}{3}\right)^5.$$

$$= 56 \left(\frac{4}{3}\right)^5$$

Ques. $(1+2x)^6 (1-x)^7$, Coefficient of $x^5 = ?$

$$(1+2x)^6 (1-x)^7$$

$$\left[\sum_{r=0}^{r=6} {}^6C_r (2x)^r\right] \times \left[\sum_{k=0}^{k=7} {}^7C_k (-1)^k x^k\right]$$

$$x^r \times x^k = x^{r+k} = x^5$$

$$\boxed{r+k=5}$$

1. $r=0$ $k=5$
2. $r=1$ $k=4$
3. $r=2$ $k=3$
4. $r=3$ $k=2$
5. $r=4$ $k=1$
6. $r=5$ $k=0$

$$\text{Coeff. of } x^5 = {}^6C_0 \cdot {}^7C_5\, 2^0 (-1)^5 + {}^6C_1\, 2^1 . {}^7C_4 (-1)^4 + {}^6C_2\, 2^2\, {}^7C_3 (-1)^3 + {}^6C_3\, 2^3\, {}^7C_2 (-1)^2 + {}^6C_4\, 2^4\, {}^7C_1 (-1)^1 + {}^6C_5\, 2^5 . {}^7C_0 (-1)^0$$

$= -21 + 420 - 2100 + 3360 - 1680 + 192$

$= 3972 - 3801$

$= 171.$

Q. Find the Coefficient of x^{53} in the expansion $\sum_{m=0}^{100} {}^{100}C_m (x-3)^{100-m} . 2^m$.

$$(ax+by)^n = \sum_{r=0}^{n} {}^nC_r\, a^{n-r} b^r x^{n-r} y^r$$

$$\sum_{m=0}^{100} {}^{100}C_m (x-3)^{100-m} \cdot 2^m = (x-3+2)^{100} = (x-1)^{100} = \sum_{r=0}^{100} {}^{100}C_r\, x^{100-r} (-1)^r$$

$$= x^{100-r} = x^{53}$$

$$= 100-53 = r \Rightarrow r = 47$$

$$r = 47$$

$$\text{Coeff. of } x^{53} = {}^{100}C_{47}(-1)^{47}$$

$$= -{}^{100}C_{47} = -{}^{100}C_{53}$$

Que (1.) Find the Coefficient of x^4 in the expansion of $(2-x+3x^2)^6$.

(2.) Find the sum of all coefficients in the expansion.

1. $$(2-x+3x^2)^6 = \sum_{r=0}^{6} {}^6C_r (2-x)^{6-r} (3x^2)^r$$

$$r = 0, 1, 2, \cancel{3}, \cancel{4}, \cancel{5}, \cancel{6}$$

$$= \sum_{r=0}^{6} {}^6C_r (2-x)^{6-r}\, 3^r x^{2r}$$

$$\text{Coeff. of } x^4 \text{ in } (2-x+3x^2)^6 = \text{Coeff. of } x^4 \left[{}^6C_0 (2-x)^6 + {}^6C_1 (2-x)^5 3^1 x^2 + {}^6C_2 (2-x)^4 3^2 x^4 \right]$$

$$= {}^6C_0 \times {}^6C_4\, 2^{6-4} (-1)^4 + {}^6C_1 \times {}^5C_2\, 2^{5-2} (-1)^2 3^1 + {}^6C_2 \cdot {}^4C_0\, 2^{4-0} (-1)^0 3^2$$

$$= 1 \times 15 \times 4 + 6 \times 10 \times 8 \times 3 + 15 \times 1 \times 16 \times 9$$

$$= 60 + 1440 + 2160$$

$$= 3660.$$

(2.) जब भी Sum of all Coefficients nikalna ho Put. $x = 1$.

Sum of all coeff.

$$= (2-1+3)^6 = 4^6.$$

Illustration 13:

In the expansion of $1+(1+x)+(1+x)^2+\dots+(1+x)^n$, find the Coefficient of x^k $(0\le k\le n)$.

$$= \frac{(1+x)^{n}+1-1}{\cancel{1}+x-\cancel{1}}$$

$$\frac{1}{x}\left((1+x)^{n+1}-1\right)$$

$$\frac{(1+x)^{n+1}}{x}-\frac{1}{x} \quad \therefore \Rightarrow {}^{n+1}C_{k+1}\,x^k.$$

Illustration -15:

Find the value of $\sum_{r=0}^{10}(-1)^r\,{}^{10}C_r\left[\frac{1}{2^r}+\frac{3^r}{2^{2r}}+\frac{7^r}{2^{3r}}+\frac{15^r}{2^{4r}}+\dots\right]$.

$$\sum_{r=0}^{10}{}^{10}C_r\left(\frac{-1}{2}\right)^r 1^{10-r}+\sum_{r=0}^{10}{}^{10}C_r\left(\frac{-3}{4}\right)^r 1^{10-r}$$

$$=\left(1-\frac{1}{2}\right)^{10}+\left(1-\frac{3}{4}\right)^{10}+\left(1-\frac{7}{8}\right)^{10}+\dots$$

$$=\left(\frac{1}{2}\right)^{10}+\left(\frac{1}{4}\right)^{10}+\left(\frac{1}{8}\right)^{10}+\dots$$

$$=\left(\frac{1}{2^{10}}\right)^1+\left(\frac{1}{2^{10}}\right)^2+\left(\frac{1}{2^{10}}\right)^3+\dots$$

$$=\frac{1/2^{10}}{1-1/2^{10}}=\frac{1}{2^{10}-1}=\frac{1}{1023}.$$

Multinomial Expansion :

$n \in I^+$

$$(x_1+x_2+\ldots\ldots+x_r)^n = \sum \frac{n!}{P_1!\,P_2!\,P_3!\ldots P_r!} x_1^{P_1} x_2^{P_2} \ldots\ldots x_r^{P_r}$$

→ $0 \le P_1, P_2, \ldots\ldots P_r \le n$, $P_1, P_2, \ldots\ldots, P_r \in W$,

→ $P_1 + P_2 + \ldots\ldots + P_r = n$

→ No. of solution set $(P_1, P_2, \ldots\ldots, P_r)$ = no. of term the above Expansion.

$= {}^{n+r-1}C_{r-1}$.

Ex: $(x+y+z)^{10} = \sum \frac{10!}{a!\,b!\,c!} x^a y^b z^c$

→ No. of term $= {}^{10+3-1}C_{3-1} = {}^{12}C_2 = \frac{12\times 11}{2} = 66$

$a+b+c=10$

→ $a=0,\ b=0,\ c=10$ / $a=0,\ b=1,\ c=9$

$$(x+y+z)^{10} = \underbrace{\frac{10!}{0!\,0!\,10!} x^0 y^0 z^{10} + \frac{10!}{0!\,1!\,9!} x^0 y^1 z^9 + \ldots\ldots + \frac{10!}{10!\,0!\,0!} x^{10} y^0 z^0}_{66 \text{ terms.}}$$

Que. find Coeff. of $a^3 b^4 c^1$ in Expansion of $(1+a-b+c)^9$

$$= (1+a-b+c)^9 = \sum \frac{9!}{P_1!\,P_2!\,P_3!\,P_4!} 1^{P_1} a^{P_2} (-b)^{P_3} c^{P_4}$$

$P_2 = 3$

$P_3 = 4$

$P_4 = 1$

$P_1 = 1$

$\therefore$ Coeff of $a^3 b^4 c^1 = \frac{9!}{1!\,3!\,4!\,1!}$

$= \frac{9!}{3!\,4!} = 2520$

Ques. Find the Sum:

1. ${}^nC_0 - {}^nC_2 + {}^nC_4 - {}^nC_6 + \cdots$

$(1+x)^n = {}^nC_0 + {}^nC_1 x^1 + {}^nC_2 x^2 + {}^nC_3 x^3 + {}^nC_4 x^4 + \cdots + {}^nC_n x^n$

if $x = i$ $\quad i^2 = -1$, $i^3 = -i$, $i^4 = 1$

$(1+i)^n = {}^nC_0 + {}^nC_1 i - {}^nC_2 - {}^nC_3 i + {}^nC_4 + {}^nC_5 i - {}^nC_6 - {}^nC_7 i + {}^nC_8 \cdots$

$= ({}^nC_0 - {}^nC_2 + {}^nC_4 - {}^nC_6 + {}^nC_8 - {}^nC_{10} \cdots) + i({}^nC_1 - {}^nC_3 + {}^nC_5 - {}^nC_7 + \cdots)$

$\boxed{(Cis\theta)^n = Cis\, n\theta}$

$1 + i \Rightarrow \sqrt{2}\left(\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}\right)$

$\sqrt{2}(\cos\pi/4 + i\sin\pi/4)$

$\boxed{1 + i \Rightarrow \sqrt{2}\, cis(\pi/4)}$

$(\sqrt{2}\, cis\, \pi/4)^n = ({}^nC_0 \cdot {}^nC_2 + {}^nC_4 - {}^nC_6 \cdots) + i({}^nC_1 - {}^nC_3 + {}^nC_5 - {}^nC_7 + \cdots)$

$2^{n/2}\, cis\left(\frac{n\pi}{4}\right) = ({}^nC_0 - {}^nC_2 + {}^nC_4 - {}^nC_6 \cdots) + i({}^nC_1 - {}^nC_3 + {}^nC_5 - {}^nC_7 \cdots)$

$= 2^{n/2}\cos\frac{n\pi}{4} + i\, 2^{n/2}\sin\left(\frac{n\pi}{4}\right)$

${}^nC_0 - {}^nC_2 + {}^nC_4 - {}^nC_6 + \cdots = 2^{n/2}\cos\frac{n\pi}{4}$

${}^nC_1 - {}^nC_3 + {}^nC_5 - {}^nC_7 + \cdots = 2^{n/2}\sin\frac{n\pi}{4}$

Ques. Prove: ${}^{100}C_0 + {}^{100}C_3 + {}^{100}C_6 + \cdots = \frac{1}{3}(2^{100} - 1)$

$\longrightarrow 1 + x + x^2 = 0$

$x = \omega, \omega^2$

Imp.

$1 + \omega + \omega^2 = 0$ | $\omega = \frac{-1+\sqrt{3}i}{2}$, $\omega^2 = \frac{-1-\sqrt{3}i}{2}$ | $\omega^3 = 1$ | $\omega = \frac{1}{\omega^2}$ | $\omega^2 = \frac{1}{\omega}$

$(1+x)^{100} = {}^{100}C_0 + {}^{100}C_1 x + {}^{100}C_2 x^2 + \cdots + {}^{100}C_{100} x^{100}$

$x = 1$ $\quad 2^{100} = {}^{100}C_0 + {}^{100}C_1 + {}^{100}C_2 + \cdots + {}^{100}C_{100}$

$x=\omega$

$$(1+\omega)^{100} = {}^{100}C_0 + {}^{100}C_1\omega + {}^{100}C_2\omega^2 + {}^{100}C_3\cdot 1 + {}^{100}C_4\omega + {}^{100}C_5\omega^2 + \ldots + {}^{100}C_{100}\omega$$

$x=\omega^2$

$$(1+\omega^2)^{100} = {}^{100}C_0 + {}^{100}C_1\omega^2 + {}^{100}C_2\omega + {}^{100}C_3 1 + {}^{100}C_4\omega^2 + {}^{100}C_5\omega + \ldots + {}^{100}C_{100}\omega^2$$

$$2^{100} + (-\omega^2)^{100} + (-\omega)^{100} = 3\left({}^{100}C_0\right) + \underbrace{{}^{100}C_1\left(1+\omega+\omega^2\right)}_{0} + {}^{100}C_2\left(1+\omega^2+\omega\right) + {}^{100}C_3\left(1+1+1\right) +$$
$$ {}^{100}C_4\left(1+\omega+\omega^2\right) + {}^{100}C_5\left(1+\omega^2+\omega\right) + \ldots + {}^{100}C_{100}\left(1+\omega+\omega^2\right)$$

$$2^{100} + \omega^2 + \omega = 3\left({}^{100}C_0 + {}^{100}C_3 + {}^{100}C_6 + \ldots\ldots\right)$$

$$\Rightarrow 2^{100} + \omega^2 + \omega = 3\left({}^{100}C_0 + {}^{100}C_3 + {}^{100}C_6 + \ldots\ldots\right)$$

$$\Rightarrow \left({}^{100}C_0 + {}^{100}C_3 + {}^{100}C_6 + \ldots\right) \Rightarrow \frac{1}{3}\left(2^{100}-1\right)$$

Que. CAE-3.

3. If $({}^nC_0+{}^nC_1)({}^nC_1+{}^nC_2)\ldots\ldots({}^nC_{n-1}+{}^nC_n) = k\frac{(n+1)^n}{n!}$, find value of k.

$\ldots\ldots({}^nC_{n-1}+{}^nC_n) = k\frac{(n+1)^n}{n!}$

$= ({}^nC_0+{}^nC_1)({}^nC_1+{}^nC_2)\ldots\ldots$

$\because {}^nC_r + {}^nC_{r+1} = {}^{n+1}C_{r+1}$

LHS.

$$= \left({}^{n+1}C_1\right)\left({}^{n+1}C_2\right)\ldots\ldots\left({}^{n+1}C_n\right)$$

$$= \frac{(n+1)!}{n!\,1!}\cdot\frac{(n+1)!}{(n-1)!\,2!}\ldots\ldots\frac{(n+1)!}{1!\,n!}$$

$$= (n+1)^n\left(\frac{n!}{n!\,1!}\cdot\frac{n!}{(n-1)!2!}\cdot\frac{n!}{(n-2)!3!}\ldots\ldots\frac{n!}{1!n!}\right)$$

$$= \frac{(n+1)^n}{(n(n+1)(n-2)\ldots\ldots 1}\left(\frac{n!}{(n-1)!1!}\cdot\frac{n!}{(n-2)!2!}\cdot\frac{n!}{(n-3)!3!}\ldots\ldots\frac{n!}{0!\,n!}\right)$$

$$= \frac{(n+1)^n}{n!}\left({}^nC_1\cdot{}^nC_2\cdot{}^nC_3\ldots\ldots{}^nC_n\right)$$

4. If $(1+x+x^2) = a_0 + a_1 x + a_2 x^2 + \dots + a_{2n} x^{2n}$, find values of $a_0 + a_3 + a_6 + \dots$

Hint

$$= \quad x = 1$$
$$x = \omega$$
$$+ \quad x = \omega^2$$

$$3(a_0 + a_3 + a_6 \dots + \dots = (1+1+1)^n + (1+\omega+\omega^2) + (1+\omega^2+\omega^4)^n$$
$$= 3^n + 0 + 0$$

$$3(a_0 + a_3 + a_6 + \dots) = 3^n$$

5. Find the Coefficient of x^{50} in $\underbrace{(x+{}^{101}C_0)(x+{}^{101}C_1)\dots(x+{}^{101}C_{50})}_{51 \text{ factors}}$.

$$(x+{}^{101}C_0)(x+{}^{101}C_0)\dots(x+{}^{101}C_{50}) = 1x^{51} + \underbrace{({}^{100}C_0 + {}^{101}C_1 + \dots + {}^{101}C_{50})x^{50}}_{\Downarrow} + \dots 1 \dots$$

Coeff. of x^{50}

$$= {}^{101}C_0 + {}^{101}C_1 + \dots + {}^{101}C_{50}$$

$$= \frac{2^{101}}{2} = 2^{100}$$

6. Find value of ${}^{100}C_0 + {}^{100}C_4 + {}^{100}C_8 + \dots + {}^{100}C_{100}$

$$= (1+x)^{100} = 100C_0 + 100C_1 x + {}^{100}C_2 x^2 + {}^{100}C_3 x^3 + 100C_4 x^4 + 100C_5 x^5 + {}^{100}C_6 x^6$$
$$+ 100C_7 x^7 + {}^{100}C_8 x^8 + \dots$$

$$x=1 \quad 2^{100} = {}^{100}C_0 + {}^{100}C_1 + {}^{100}C_2 + {}^{100}C_3 + {}^{100}C_4 + {}^{100}C_5 + {}^{100}C_6 + {}^{100}C_7 + {}^{100}C_8 + \dots$$

$$x=-1 \quad 0 = {}^{100}C_0 - {}^{100}C_1 + {}^{100}C_2 - {}^{100}C_3 + {}^{100}C_4 - {}^{100}C_5 + {}^{100}C_6 - {}^{100}C_7 + {}^{100}C_8 + \dots$$

$$x=i \quad (1+i)^{100} = {}^{100}C_0 + {}^{100}C_1 i - 100C_2 - {}^{100}C_3 i + {}^{100}C_4 + {}^{100}C_5 i - 100C_6 - {}^{100}C_7 i + \dots$$

$$x=-i \quad (1-i)^{100} = {}^{100}C_0 - {}^{100}C_1 i - {}^{100}C_2 + {}^{100}C_3 i + {}^{100}C_4 - {}^{100}C_5 i - 100C_6 + {}^{100}C_7 i + {}^{100}C_8 + \dots$$

$$2^{100} + (1+i)^{100} + (1-i)^{100} = 4\left[{}^{100}C_0 + {}^{100}C_4 + {}^{100}C_8 + \dots + {}^{100}C_{100}\right]$$

$$= 2^{100} + \left(\sqrt{2}\,cis\,{}^{\pi}/_{4}\right)^{100} + \left(\sqrt{2}\,cis\left(-{}^{\pi}/_{4}\right)\right)^{100}$$

$$= 2^{100} + 2^{50}\left[\cos 25\pi + i\cancel{\sin 25}\pi + \cos(25\pi) - i\cancel{\sin(25\pi)}\right]$$

$$= 2^{100} + 2^{51}(-1) = \frac{2^{100} - 2^{51}}{4}$$

Note:

$$2 \leq \left(1 + {}^{1}/_{n}\right)^{n} < 3 \qquad n \geq 1,\ n \in N.$$

Proof: $\left(1 + {}^{1}/_{n}\right)^{n} = {}^{n}C_0 + {}^{n}C_1 \cdot {}^{1}/_{n} + {}^{n}C_2\ {}^{1}/_{n^2} + \cdots\cdots + {}^{n}C_n \frac{1}{n^n}.$

$$= 1 + \frac{n}{n} + \frac{n(n-1)}{2!} \times {}^{1}/_{n^2} + \frac{n(n-2)(n-3)}{3!}\ {}^{1}/_{n^3} + \cdots\cdots + \frac{n(n-1)(n-2)\cdots(n-n)}{0!} \cdot {}^{1}/_{n^2}$$

$$\left(1 + {}^{1}/_{n}\right)^{n} = 1 + 1 + \frac{1}{2!}(1)\left(1 - {}^{1}/_{n}\right) + {}^{1}/_{3!}\left(1 - {}^{2}/_{n}\right)\left(1 - {}^{3}/_{n}\right) + \cdots\cdots$$

$$\therefore\ 2 \leq \left(1 + {}^{1}/_{n}\right)^{n}$$

$$\left(1 + {}^{1}/_{n}\right)^{n} < 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \cdots\cdot$$

$$\left(1 + {}^{1}/_{n}\right)^{n} < 1 + 1 + \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \cdots\cdot$$

$$\left(1 + {}^{1}/_{n}\right)^{n} < 1 + \frac{1}{{}^{1}/_{2}}$$

$$\left(1 + {}^{1}/_{n}\right)^{n} < 3$$

Paper Discussion -(22|02|26)

1. The value of $(2!P_0 - 3.{}^2P_1 + 4.{}^3P_2 \ldots\ldots$ up to 51^{th} term$)$ + $(1! - 2! + 3! - \ldots\ldots$ up to 51^{th} term$)$ is equal to :

$(2.{}^1P_0 - 3.{}^2P_1 + 4.{}^3P_2 \ldots\ldots$ up to $51^{th}) + (1! - 2! + 3! - 4! \ldots - 50! + 51!)$

$[2.{}^1P_0 - 3\,{}^2P_1 + 4.{}^3P_2 \ldots\ldots + 52\ {}^{51}P_{50}) + (1! - 2! + 3! - 4! + \ldots\ldots 50! + 51!)$

$(2.1! - 3.2! + 4.3! - 5.4! \ldots\ldots + 52.51!) + (1! - 2! + 3! - 4! + \ldots\ldots 50! + 51!)$

$(\cancel{2!} - \cancel{3!} + \cancel{4!} - \cancel{5!} \ldots\ldots + 52!) + (1 - \cancel{2!} + \cancel{3!} - \ldots\ldots + 51!)$

$= 1 + 52!$ —— (b).

3. If the number of five digit number with distinct digits are 2 at the 10^{th} place is 336 k, then k is to.

0, 1, 2, 3, 4, 5, 6, 7, 8, 9

$$\underbrace{\_\ \ \_\ \ \_\ \ 2\ \ \_}_{\text{5 digit No.}}$$

No. of number $= 8 \times 8 \times 7 \times 1 \times 6$

$= 42 \times 8 \times 8$

$= 336 \times 8$

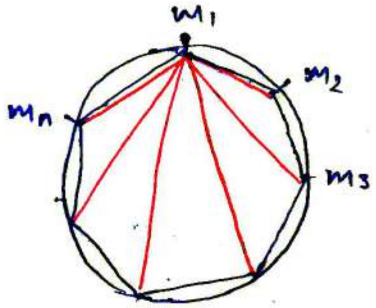
$= 366 \times k$

$\boxed{k = 8}$

6.

$n > 2$

n metro stations



→ Red line $\quad {}^{n}C_2 - n$

→ Blue line $\quad n$

$${}^{n}C_2 - n = 99n$$

$${}^{n}C_2 = 100n$$

$$\frac{n(n-1)}{2} = 100n$$

$$\frac{n(n-1) - 200n}{2} = 0$$

$$n[n-1-200] = 0$$

$$n \neq 0, \; n = 201.$$

7. The Value of $\sum_{r=0}^{20} {}^{50-r}C_6$ is equal to:

$$\sum_{r=0}^{20} {}^{50-r}C_6 = {}^{50}C_6 + {}^{49}C_6 + {}^{48}C_6 + \cdots\cdots + \overbrace{{}^{31}C_6 + \overbrace{{}^{30}C_6 + {}^{30}C_7}^{{}^{31}C_7}}^{{}^{32}C_7} - {}^{30}C_7$$

$${}^{n}C_r = {}^{n}C_{n-r} \;\Big|\; {}^{n}C_r + {}^{n}C_{r-1} = {}^{n+1}C_r$$

$$= {}^{51}C_7 - {}^{30}C_7$$

11. If $\{P\}$ denotes the fractional part of the number P, then $\left\{\frac{3^{200}}{8}\right\}$, is equal to.

$$= \left\{\frac{3^{200}}{8}\right\} = \left\{\frac{(3^2)^{100}}{8}\right\}$$

$$= \left\{\frac{(1+8)^{100}}{8}\right\}$$

$$= \left\{\frac{{}^{100}C_0 + {}^{100}C_1 8^1 + {}^{100}C_2 8^2 + \cdots\cdots + {}^{100}C_{100} 8^{100}}{8}\right\}$$

$$= \left\{\frac{1}{8}\right\} = \frac{1}{8}$$

⑫. If ${}^{20}C_1 + (2^2)\,{}^{20}C_2 + (3^2)\,{}^{20}C_3 + \cdots + (20^2)\,{}^{20}C_{20} = A(2^{\beta})$, then the Ordered Pair (A, β) is equal to.

$$0^2\,{}^{20}C_0 + 1^2\,{}^{20}C_1 + (2^2)\,{}^{20}C_2 + (3^2)\,{}^{20}C_3 + \cdots + 20^2\,{}^{20}C_{20}$$

$$\sum_{r=0}^{20} r^2\,{}^{20}C_r \quad \Big| \quad (1+x)^{20} = {}^{20}C_0 + {}^{20}C_1x^1 + {}^{20}C_2x^2 + \cdots + 20.\,{}^{20}C_{20}x^{20}$$

$$20(1+x)^{19} = 0 + {}^{20}C_1 + 2.\,{}^{20}C_2x^1 + \cdots + 20.\,{}^{20}C_{20}x^{19}$$

$$20(1+x)^{19}x = 0\,x + {}^{20}C_1x + 2.\,{}^{20}C_2x^2 + \cdots + 20.\,{}^{20}C_{20}x^{20}$$

diff w.r.t x

$$20(1+x)^{19} + 380(1+x)^{18}\cancel{x} = 0 + 1^2\,{}^{20}C_1 + 2^2\,{}^{20}C_2x + 3^2\,{}^{20}C_3x^2 + \cdots + 20^2\,{}^{20}C_{20}x^{n}$$

$x = 1$

$$20\times2^{19} + 380\times2^{18} = 1^2\,{}^{20}C_1 + 2^2\,{}^{20}C_2 + \cdots + 20^2\,{}^{20}C_{20}$$

$2^{18}[420]$

$A\cdot2^{\beta}$

$A = 420$
$\beta = 18$

14. If $\sum_{r=0}^{25}\{{}^{50}C_r\cdot{}^{50-r}C_{25-r}\} = k({}^{50}C_{25})$, then <u>k</u> is equal to:

$$= \sum_{r=0}^{25}\{{}^{50}C_r\cdot{}^{50-r}C_{25-r}\} = k\left({}^{50}C_{25}\right)$$

$$= \cancel{\sum_{r=0}^{25}}\ {}^{50}C_0\cdot{}^{50}C_{25} + {}^{50}C_1\cdot{}^{49}C_{24} + {}^{50}C_2\cdot{}^{48}C_{23} \cdots + {}^{50}C_{25}\cdot{}^{25}C_0$$

$$= \text{Coeff of } x^{25} \text{ in } \Big({}^{50}C_0(1+x)^{50} + {}^{50}C_1(1+x)^{49} + {}^{50}C_2(1+x)^{48} + \cdots + {}^{50}C_{25}(1+x)^{25} + {}^{50}C_{26}(1+x)^{24} + \cdots + {}^{50}C_{50}(1+x)^{0}\Big]$$

Coeff. of x^{25} in $\left((1+x+1)^{50}\right)$.

Coeff. of x^{25} in $(2+x)^{50} = {}^{50}C_{25}\ 2^{50-25}$.

$$= 2^{25}.{}^{50}C_{25}.$$

$$\Rightarrow k = 2^{25}.$$

15.

$= (x+10)^{50} + (x-10)^{50} = a_0 + a_1 x + a_2 x^2 + \ldots\ldots + a_{50} x^{50}$

$\sum_{r=0}^{50} {}^{50}C_r\ x^{50-r}\ 10^r + \sum_{r=0}^{50} {}^{50}C_r\ x^{50-r}\ 10^r (-1)^r$

$\frac{a_r}{a_r} = ?$ $\quad a_2 =$ Coeff. of $x^2 = {}^{50}C_{48}\ 10^{48} + {}^{50}C_{48}\ 10^{48} = 2.\ {}^{50}C_{48}\ 10^{48}$

$a_0 =$ Coeff of $x^0 = {}^{50}C_{50}\ 10^{50} + {}^{50}C_{50}\ 10^{50} = 2 \times {}^{50}C_{50}\ 10^{50}$

$$\frac{a_2}{a_0} = \frac{\cancel{2} \times {}^{50}C_{48}\ 10^{48}}{\cancel{2} \times {}^{50}C_{50} \times 10^{50}} = \frac{\cancel{50} \times 49}{2 \times 1 \times \underset{2}{\cancel{10^2}}}. = \frac{49}{4} = 12.25.$$

16.

$= \alpha > 0,\ \beta > 0$

$\alpha^2 + \beta^2 = 4$

$\left(\alpha . x^{1/9} + \beta x^{-1/6}\right)^{10} = ?$

$$T_{r+1} = {}^{10}C_r \, \alpha^{10-r} \, x^{\frac{10-r}{9}} \, \beta^r \, x^{-r/6}$$

Coeff. of x^0

$$x^{\frac{10-r}{9} - \frac{r}{6}} = x^0$$

$$\frac{10-r}{9} - \frac{r}{6} = 10$$

$$60 - 6r - 9r = 0$$

$$\boxed{r = 4}$$

$$T_5 = {}^{10}C_4 \, \alpha^6 \beta^4$$

$$= 10C_4 \left(\alpha^3 \beta^2\right)^2$$

$$\alpha^3 + \beta^2 = 4$$

$$\frac{\alpha^2 + \beta^2}{2} \geq \left(\alpha^3 \beta^2\right)^{\frac{1}{2}} \qquad AM \geq GM$$

$$\left(\frac{4}{2}\right)^2 \geq \alpha^3 \beta^2$$

$$4 \geq \alpha^3 \beta^2$$

$$(T_5)_{max} = {}^{10}C_4 \; 9^2$$

$$= 16 \cdot {}^{10}C_4$$

$$= 16 \times 210$$

$$= 336 \times 10$$

$$\downarrow$$

$$K.$$

$= T_{r+1} = {}^{n}C_{r}\ 3^{\frac{n-r}{2}}\ 5^{r/8}$

$n = 2m$.

r	$\frac{n-r}{2}$	$r/8$
0	$m-0$	✓
8	$m-4$	✓
16	$m-8$	✓
24	$m-12$	

$T_1 = 0$

$T_2 = 4$

$T_3 = 8$

$T_4 = 12$

$\vdots$

$T_{33} = ?$

$T_{33} = 0 + 4(33-1)$

$= 128$

$m_{\min} = 128$

$n = 2m = 2 \times 128$

$= 256$.

22. The total number of 3-digit numbers, whose sum of digit is 10 is ——

${}^{9+3-1}C_{3-1}$ (54)

$= {}^{11}C_2$ (55)

$a+b+c = 10$

$0 \le b, c \le 9, \quad 1 \le a \le 9$

$1 \le a \le 9$

$0 \le a-1 \le 8$

$a' = a-1$

$a' = 9$

$\boxed{a = 10}$

$(10, 0, 0)$

$(a-1) + b + c = 9$

$$\left.\begin{array}{l} a' + b + c = 9 \\ 0 \le a', b, c \le 9 \end{array}\right\} \quad {}^{9+3-1}C_{3-1} \quad {}^{11}C_2 = 55$$

$\downarrow$

(10,0,0) is included in soln $\Rightarrow 55 - 1 = 54$.

: Problems based on Remainder :

Q. Find remainder when a^n is divided by b. $a, b \in I^+$

↳ Basic Approach.

↳ using Binomial theorem.

$$\Rightarrow \left\{\frac{a^n}{b}\right\} = \text{fraction}\left(\frac{c}{b}\right) \quad c < b \text{ , } c \rightarrow \text{some integer}$$

$$= \frac{c}{b}$$

$$a^n = b \times \underset{\text{integer}}{\underset{\downarrow}{\boxed{q_1}}} + \boxed{c} \text{ integer}$$

We used to write $a = b \times k + 1$ or $b \times k - 1$

if $a = bk + 1$

$$a^n = (bk+1)^n$$

$$= {}^nC_0 (bk)^n + {}^nC_1 (bk)^{n-1} + {}^nC_2\, b^{n-2} k^{n-2} + \dots + {}^nC_{n-1} b^1 k^1 + \boxed{{}^nC_n b^0 k^0}$$

$$a^n = b\left({}^nC_0 b^{n-1} k^n + {}^nC_1 b^{n-2} k^{n-1} + \dots + {}^nC_{n-1} k^1\right) + 1$$

$$a^n = bq_1 + 1$$

q_1 = Some integer

$$\therefore a^n = bq_1 + c$$

$$a^n = bq_1 + 1$$

$$\therefore \boxed{c = 1}$$

SPS remainder technique:

$R\left(\frac{a^n}{b}\right)$ = represent remainder when b divides a^n.

आपको a को b से divide करने की कोशिश करना है और Get(r).

$$b\overline{)\,a\,(}\quad \text{if } r_1 = 1 \therefore R\left(\frac{a^n}{b}\right) = 1$$
$$\underline{\;\;r_1\;\;}\quad \text{if } r_1 \neq 1 \therefore R\left(\frac{a^n}{b}\right) = R\left(\frac{r_1^{\,n}}{b}\right)$$

Or,

अगर $a < b$

पहले n से कुछ Value लेकर a को देंगे जिससे $a^{\text{something}} > b$ then divide $a^{\text{something}}$ by b & then repeat the same Process

Ques. $\frac{2^{200}}{7}$ find remainder.

method – ①

2^{200} divided by 7

$7k+1$

or

$7k-1$

$P = 2^{200}$

$P = 2^2 \cdot 2^{198}$

$P = 4 \cdot (2^3)^{66} = 4 \cdot (8)^{66}$

$P = 4(1+7)^{66}$

$$P = 4\left({}^{66}C_0 + \underbrace{{}^{66}C_1 7 + {}^{66}C_2 7^2 + \dots\dots + {}^{66}C_{66} 7^{66}}_{\Downarrow}\right)$$

$P = 4(1+7k) \qquad k \in I^+$

$2^{200} = 4 + 7 \times 4 + k$

$2^{200} = 7 + 28k$

$= 7q + \textcircled{4}$
$\downarrow$
remainder

method – ⑪

$$R\left(\frac{2^{200}}{7}\right) = R\left(\frac{2^2 \cdot 2^{198}}{7}\right)$$
$$= R\left(\frac{4 \cdot 8^{66}}{7}\right)$$
$$= R\left(\frac{4 \cdot 1^{66}}{7}\right)$$
$$= R\left(\frac{4}{7}\right) = 4.$$

Q. $\frac{25^{999}}{21}$. (find remainder)

$$R\left(\frac{25^{999}}{21}\right)$$

$$= R\left(\frac{4^{999}}{21}\right)$$

$$= R\left(\frac{(4^3)^{333}}{21}\right)$$

$$= R\left(\frac{64^{333}}{21}\right)$$

$$= 64 = 21 \times 3 + 1$$

$$= R\left(\frac{1^{333}}{21}\right)$$

$$= \boxed{1.}$$

Q. $R\left(\frac{2222^{2222}}{7}\right)$

$$= R\left(\frac{3^{2222}}{7}\right)$$

$$= R\left(\frac{9^{1111}}{7}\right)$$

$$= R\left(\frac{2^{1111}}{7}\right)$$

$$= R\left(\frac{2 \cdot 2^{1110}}{7}\right)$$

$$= R\left(\frac{2 \cdot (2^3)^{370}}{7}\right)$$

$$= R\left(\frac{2 \cdot 1^{370}}{7}\right) = \boxed{2}.$$

Ques. $25^{13} + 3^{13}$ when divisible by 7. what is remainder

method – ①

$$R\left(\frac{25^{13}}{7}\right) = R\left(\frac{4^{13}}{7}\right)$$

$$= R\left(\frac{4 \cdot (4^3)^4}{7}\right)$$

$$= R\left(\frac{4 \cdot 1^4}{7}\right)$$

$$= 4$$

Ques. The sum of last two digits of the number 7^{101} is $\underline{0+7}=7$

$$7(7^{100}) = 7(7^2)^{50}$$
$$= 7(49)^{50} = 7(50-1)^{50}$$
$$= 7\left({}^{50}C_0\,50^{50} - {}^{50}C_1\,50^{49} + {}^{50}C_2\,50^{48} - {}^{50}C_3\,50^{47} + \dots + {}^{50}C_{48}\,50^{2} - {}^{50}C_{49}\,50^{1} + {}^{50}C_{50}\right)$$
$$= 7\left(000 + \frac{\overset{25}{\cancel{50}}\times 49}{\cancel{2}} \times 2500 - 50\times 50 + 1\right)$$
$$= 7(\quad 25^2 \times 4900 - 2500 + 1$$
$$= 7(\quad 2500(25\times 49 - 1) + 1) \Rightarrow 7(..2500\times 1224 + 1) = 7(.001)$$

Ques: Prove that $\left[(7+3\sqrt{5})^{25}\right]$ is an odd number. $[\cdot] \longrightarrow$ G.I.F.

$$= \left[(7+3\sqrt{5})^{25}\right] = \text{integral Part.}$$

$$N = (7+3\sqrt{5})^{25} = \underset{\substack{\uparrow \\ \text{integral Part}}}{I} + f \longrightarrow \text{fractional part.}$$

$$I = \left[(7+3\sqrt{5})^{25}\right] \hookrightarrow \left(\begin{array}{l}\text{we have to prove} \\ \text{'I' is an odd No.}\end{array}\right)$$

$$f = \{(7+3\sqrt{5})^{25}\}$$

$$N' = (7-3\sqrt{5})^{25} = f' \quad (\text{A new fraction}).$$

$$N + N' = \underbrace{\underbrace{(7+3\sqrt{5})^{25} +}_{2(\ )} (7-3\sqrt{5})^{25}}_{\text{even No.}} = I + f + f'$$

$$I + \underbrace{f + f'}_{\text{integer}} = \text{even no. (Integer)}$$

$$0 < \underbrace{f + f'}_{\text{integer}} < 2$$

$$\therefore f + f' = 1 \quad (\text{odd No.})$$

$$I + \text{odd no.} = \text{even no.}$$

$$\therefore I = \text{odd no.}$$

Hence Proved.

CAE - 3.

1. Find the approximation of $(0.99)^5$ using the first three terms of its expansion.

$(0.99)^5$

$(1-0.01)^5$

$$\left(1-\frac{1}{100}\right)^5 = \underbrace{{}^5C_0 - {}^5C_1\left(\frac{1}{100}\right) + {}^5C_2\left(\frac{1}{100}\right)^2}_{\downarrow} - \underbrace{{}^5C_3\left(\frac{1}{100}\right)^3 + {}^5C_4\left(\frac{1}{100}\right)^4 - {}^5C_5\left(\frac{1}{100}\right)^5}_{\text{ignore}}$$

$$= 1 - \frac{5}{100} + \frac{10}{10000} \text{ approx.}$$

$$= 1 - 0.05 + 0.001$$

$$= 0.95 + 0.001$$

$$= 0.951$$

2. If a and b are distinct integers, Prove that $a-b$ is a factor of $a^n - b^n$, whenever n is a positive integer.

Mathematical Induction.

$n=1$

$P(n)\ P = a^n - b^n$

$n=1$

$P(1)\ P = a - b = (a-b) \times 1 + 0$ —①

∴ so divisible by $a-b$

let $n=k$

$P(k)P = a^k - b^k = (a-b)q_1$ —② (suppose)

Now $n=k+1$

$P(k+1)P = \underbrace{a^{k+1} - b^{k+1}}$ ~~(i) × (ii)~~ $(a+b) \times (ii)$

$$(a+b)(a^k - b^k) = (a^2-b^2)q_1$$

$$a^{k+1} - ab^k + ba^k - b^{k+1} = (a^2-b^2)q_1$$

$$a^{k+1} - b^{k+1} = (a^2-b^2)q_1 + ab^k - ba^k.$$

$$= (a-b)(a-b)q_1 + ab^k - b[(a-b)q_1 + b^k]$$

$$= (a-b)(a+b)q_1 + ab^k - b(a-b)q_1 - b^{k+1}$$

$$= (a-b)(a+b)q_1 + ab^k - b^{k+1} - b(a-b)q_1$$

$$= (a-b)[(a+b)q_1 + b^k - bq_1)$$

$$= (a-b)[aq_1 + b^k]$$

method — ②

$$P(n) = a^n - b^n$$

$$(a-b) \mid P(n)$$

$$P(n) = a^n - b^n = 0 \mid Mod\,(a-b)$$

10 मिनटे आ रहे ?

Proof: $P(n) = a^n - b^n$

$$k = a-b$$

$$k + b = a$$

$$P(n) = (k+b)^n - b^n$$

$$= \underbrace{{}^nC_0 k^n + {}^nC_1 k^{n-1} b^1 + {}^nC_2 k^{n-2} b^2 + \dots + {}^nC_{n-1} k b^{n-1}} + \underbrace{\overset{\overset{1}{\uparrow}}{{}^nC_n} k^0 \cancel{b^n - b^n}}$$

$$= {}^nC_0 k^n + {}^nC_1 k^{n-1} b + {}^nC_2 k^{n-2} b^2 + \dots + {}^nC_{n-1} k b^{n-1}$$

$$= k\,[\underbrace{{}^nC_0 k^{n-1} + {}^nC_1 k^{n-2} b + \dots + {}^nC_{n-1} b^{n-1}}_{\Downarrow \atop m}]$$

$$P(n) = km$$

$P(n) = km$

$P(n) = a^n - b^n = (a-b)m$

$\therefore (a-b) \mid P(n)$

Hence Proved.

3. Find the last three digit of the No. $(6!+1)6!$.

$$(6!+1)\,\underset{\substack{\downarrow\\ 0\\ \text{UPD}}}{6!}$$

$721 \times 72 \times 10$

$(700+20+1)\,72 \times 10$

$504000 + 14400 + 720$

$= 120$.

Q. $(599+2)^{1999}$. Find last three digit of the No.

$(599+2)^{1999}$

$(600-1+2)^{1999}$

$$(600+1)^{1999} = \underbrace{{}^{1999}C_0 (600)^{1999}}_{0000} + \cdots + \underbrace{(600)^2 + {}^{1999}C_{1998}(600) + {}^{1999}C_{1999}(600)^0}_{\underbrace{1999 \times 600 + 1}_{\underbrace{400 \quad 1}_{401}}}$$

401 last three digit of $(601)^{1999}$

Note:

$$r\,{}^{n}C_{r} = n\,{}^{n-1}C_{r-1}$$

$${}^{n}C_{r} = \frac{n}{r}\left({}^{n-1}C_{r-1}\right)$$

$${}^{n}C_{r} = \frac{n!}{(n-r)!\,r!} = \frac{n}{r}\,\frac{(n-1)!}{(n-r)!\,(r-1)!}$$

$${}^{n}C_{r} = \frac{n}{r}\,\frac{(n-1)!}{((n-1)-(r-1))!\,(r-1)!}$$

$${}^{n}C_{r} = \frac{n}{r}\left({}^{n-1}C_{r-1}\right)$$

Pattern

$$\frac{{}^{n-1}C_{r-1}}{r} = \frac{{}^{n}C_{r}}{n}$$

$$\downarrow$$

$$n \longrightarrow n+1$$
$$r \longrightarrow r+1$$

$$\frac{{}^{n}C_{r}}{r+1} = \frac{{}^{n+1}C_{r+1}}{n+1}$$

Similarly.

$${}^{n+2}C_{r+2} = \frac{n+2}{r+2}\,{}^{n+1}C_{r+1}$$

$$= \frac{(n+2)(n+1)}{(r+2)(r+1)}\,{}^{n}C_{r}$$

$$\boxed{\frac{{}^{n}C_{r}}{(r+1)(r+2)} = \frac{{}^{n+2}C_{r+2}}{(n+1)(n+2)}}$$

Q. Find the Value of the Series $3\,{}^nC_0 - 8\,{}^nC_1 + 13\,{}^nC_2 - 18\,{}^nC_3 + \ldots$

$$3\,{}^nC_0 - 8\,{}^nC_1 + 13\,{}^nC_2 - 18\,{}^nC_3 + \cdots$$

$$T_{r+1} = (-1)^r (3+5r)\,{}^nC_r$$

$$\sum_{r=0}^{n} T_{r+1} = \sum_{r=0}^{n} (-1)^r \,(3+5r)\,{}^nC_r$$

$$= 3\underbrace{\sum_{r=0}^{n}(-1)^r\,{}^nC_r}_{\Downarrow} + 5\underbrace{\sum_{r=0}^{n}(-1)^r\,r.\,{}^nC_r}_{\Downarrow}$$

$$= 3\;f(-1) \;+\; 5\;f'(-1)$$

$$= 3\times 0 + 5(0)$$

$$= 0 .$$

Product of two binomials:

1. $({}^nC_0)^2 + ({}^nC_1)^2 + ({}^nC_2)^2 + \cdots\cdots + ({}^nC_n)^2 = {}^{2n}C_n.$

Proof:

$$f(x) = (1+x)^n = {}^nC_0 + {}^nC_1 x^1 + {}^nC_2 x^2 + \cdots\cdots + {}^nC_n x^n$$

$$f(x) = (x+1)^n = {}^nC_0 x^n + {}^nC_1 x^{n-1} + {}^nC_2 x^{n-2} + \cdots\cdots + {}^nC_n x^0$$

$$(1+x)^n(x+1)^n = \underbrace{\left(\rule{3cm}{0.4pt}\right)\times\left(\rule{3cm}{0.4pt}\right)}_{(n+1)^2 \text{ terms}}$$

$$(1+x)^{2n} = a_0 + a_1x^2 + a_2x^3 + \cdots + a_n x^n \;\rule{1.5cm}{0.4pt}\; + a_{(n+1)^2}x^{(n+1)^2}$$

Coeff. of x^n in $(1+x)^{2n}$ = Coeff of x^n

$${}^{2n}C_n = ({}^nC_0)^2 + ({}^nC_1)^2 + ({}^nC_2)^2 + \cdots\cdots + ({}^nC_n)^2$$

Hence Proved

2. ${}^mC_r + {}^mC_{r-1} \cdot {}^nC_1 + {}^mC_{r-2} \cdot {}^nC_2 + \ldots\ldots + {}^nC_r = {}^{m+n}C_r$, where $r<m$
$r<n$
$m, n, r \in I^+$

Proof:

$${}^nC_0 \cdot {}^mC_r + {}^mC_{r-1} \cdot {}^nC_1 + {}^mC_{r-2} \cdot {}^nC_2 + \ldots + {}^mC_0 \cdot {}^nC_r$$

$$\Rightarrow (1+x)^n = {}^nC_0 + {}^nC_1 x + {}^nC_2 x^2 + \ldots + {}^nC_r x^r + \ldots + {}^nC_n x^n$$

$$\Rightarrow (1+x)^m = {}^mC_0 x^0 + {}^mC_1 x^1 + {}^mC_2 x^2 + \ldots + {}^mC_{r-1} x^{r-1} + {}^mC_r x^r + {}^mC_{r+1} x^{r+1} + \ldots + {}^mC_m x^m$$

$$\Rightarrow (1+x)^{m+n} = \underbrace{(\qquad\qquad) \times (\qquad\qquad)}_{(n+1)(m+1)\text{ terms.}}$$

Coeff. of x^r in $(1+x)^{m+n}$ = Coeff. of x^r

$${}^{m+n}C_r = {}^mC_r \cdot {}^nC_0 + {}^mC_{r-1}\, {}^nC_1 + \ldots + {}^nC_r\, {}^mC_0$$

Hence Proved.

3. $$({}^nC_0)^2 - ({}^nC_1)^2 + ({}^nC_2)^2 - \ldots + (-1)^n ({}^nC_n)^2 = \begin{cases} 0 & \text{if } n\text{-odd} \\ (-1)^{n/2}\, {}^nC_{n/2} & \to \text{if } n\text{-even.} \end{cases}$$

$\Downarrow$

Coeff. of x^n in $(x+1)^n (1-x)^n$

Coeff of x^n in $(1-x^2)^n$

$\Downarrow$

$(-1)^r\, {}^nC_r\, (x^2)^r$

$(-1)^r\, {}^nC_r\, x^{2r}$

$2r = n$
$r = n/2$

n-odd	n-even
not-Possible	$r = n/2$ (Possible)

Binomial theorem in any index:

$n \in Q$ (Rational Number), $|x| < 1 \Rightarrow -1 < x < 1.$

$$(1+x)^n = 1 + nx + n(n-1)\frac{x^2}{2!} + n(n-1)(n-2)\frac{x^3}{3!} + \cdots\cdots$$

$x \longrightarrow -x$

$$(1-x)^n = 1 - nx + n(n-1)\frac{x^2}{2!} - n(n-1)(n-2)\frac{x^3}{3!} + \cdots\cdots$$

OR,

$$(a+x)^n = a^n\left(1+\left(\frac{x}{a}\right)\right)^n$$

$$(a+x)^n = a^n\left[1 + n\left(\frac{x}{a}\right) + \frac{n(n-1)}{2!}\left(\frac{x}{a}\right)^2 + \frac{n(n-1)(n-2)}{3!}\left(\frac{x}{a}\right)^3 + \cdots\cdots\right]$$

Important Expansion:

1. $(1+x)^{-1} = 1 - x + x^2 - x^3 + x^4 \cdots\cdots + (-1)^r x^r + \cdots$
2. $(1-x)^{-1} = 1 + x + x^2 + x^3 + \cdots\cdots + x^r + \cdots$
3. $(1+x)^{-2} = 1 - 2x + 3x^2 - 4x^3 + \cdots\cdots$
4. $(1-x)^{-2} = 1 + 2x + 3x^2 + 4x^3 + \cdots\cdots$

Ques. If $p+q=1$ then show that $\sum_{r=0}^{n} r^2 \cdot {}^{n}C_r\, p^r q^{(n-r)} = npq + n^2p^2.$

Proof: $= \sum_{r=0}^{n} r^{2} \frac{n}{r}\, {}^{n-1}C_{r-1}\, p^r q^{n-r}$

$\Rightarrow n \sum_{r=0}^{n} (r-1+1)\, {}^{n-1}C_{r-1}\, p^r q^{n-r}$

$\Rightarrow n \sum_{r=0}^{n} (r-1)\, {}^{n-1}C_{r-1}\, p^r q^{n-r} + n \sum_{r=0}^{n} {}^{n-1}C_{r-1}\, p^r q^{n-r}$

$$\Rightarrow n\sum_{r=0}^{n}\left(\cancel{(r-1)}\frac{n-1}{\cancel{(r-1)}}\ {}^{n-2}C_{r-2}+{}^{n-1}C_{r-1}\right)p^{r}q^{n-r}$$

$$\Rightarrow n\sum_{r=0}^{n}\left((n-1)\ {}^{n-2}C_{r-2}+{}^{n-1}C_{r-1}\right)p^{r}q^{n-r}$$

$$\Rightarrow n(n-1)\sum_{r=0}^{n}{}^{n-2}C_{r-2}\,p^{r}.\,q^{n-r}+n\sum_{r=0}^{n}{}^{n-1}C_{r-1}\,p^{r}q^{n-r}$$

$$\Rightarrow n(n-1)p^{2}\sum_{r=0}^{n}{}^{n-2}C_{r-2}\,p^{r-2}\,q^{(n-2)-(r-2)}+np\sum_{r=0}^{n}{}^{n-1}C_{r-1}\,p^{r-1}\,q^{(n-1)-(r-1)}$$

$$\Rightarrow n(n-1)p^{2}\,(q+p)^{n-2}+np(q+p)^{n-1}$$

$$\Rightarrow n^{2}p^{2}-np^{2}+np$$

$$\Rightarrow n^{2}p^{2}+np(1-p)$$

$\Rightarrow n^{2}p^{2}+npq.$ Proved.

Ques. Find value of:

$${}^{30}C_{0}\,{}^{30}C_{15}\,2^{15}-{}^{30}C_{1}\,{}^{29}C_{4}\,2^{14}+{}^{30}C_{2}\,{}^{28}C_{13}\,2^{13}-\ldots\ldots-{}^{30}C_{15}\,{}^{15}C_{0}.$$

$$\Rightarrow T_{r+1}=(-1)^{r}\,{}^{30}C_{r}\ {}^{30-r}C_{15-r}\,(2)^{15-r}$$

$$\Rightarrow T_{r+1}=(-1)^{r}\frac{30!}{(30-r)!\,r!}\times\frac{(30-r)!}{[(30-r)-(15-r)]!\,(15-r)!}\ 2^{(15-r)}$$

$$T_{r+1}=(-1)^{r}\frac{30!}{15!\,(15-r)!\,r!}\ 2^{15-r}$$

$$T_{r+1}=\frac{30!}{(15!)^{2}}\ \frac{15!}{(15-r)!\,r!}\ (-1)^{r}\ 2^{15-r}$$

$$\sum_{r=0}^{15}T_{r+1}=\frac{30!}{(15!)^{2}}\underbrace{\sum_{r=0}^{15}{}^{15}C_{r}\,(-1)^{r}\,2^{15-r}}_{}\quad=\frac{30!}{15!\,15!}\,(2-1)^{15}={}^{30}C_{15}.$$

The End